

Broken symmetry can yield a positive effective G in conformal gravity

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We modify the action of Mannheim's conformally invariant model by changing the sign of two coefficients. This breaks conformal symmetry, but results in a cosmology that has a positive effective G and at the same time retains one of the main advantages of the Mannheim model, a possible solution of the cosmological constant problem.

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I. INTRODUCTION

In a series of articles following his seminal paper of 1989 [1], Mannheim has developed an alternate theory of gravitation called conformal gravity. We will be concerned here with the application of this theory to cosmology; see the review [2], referred to as PM from now on. The field equations are (PM, equation (186)):

$$4\alpha_g W^{\mu\nu} = T^{\mu\nu} \quad (1)$$

where $W^{\mu\nu}$ is the Weyl conformal tensor (PM, equations (185), (107) and (108)), and α_g is a dimensionless coupling constant. $T^{\mu\nu}$ is the energy-momentum tensor, defined in such a way that the field equations are invariant under conformal transformations (PM, equation (176)):

$$g_{\mu\nu} \rightarrow e^{2\alpha(x)} g_{\mu\nu} \quad (2)$$

Mannheim includes a fermion field ψ and a scalar field S , and writes the action as (PM, equation (61)):

$$I_M = - \int d^4x (-g)^{1/2} \left\{ \frac{1}{2} S^{;\mu} S_{;\mu} - \frac{1}{12} S^2 R^\mu{}_\mu + \lambda S^4 + i\bar{\psi}\gamma^\mu(x) [\partial_\mu + \Gamma_\mu(x)] \psi - h S \bar{\psi}\psi \right\} \quad (3)$$

We use the notation of Mannheim: metric signature is $(-+++)$, $g = +\text{Det}(g_{\mu\nu})$, and the curvature tensors are defined as in Weinberg [3]. We set $c = 1$ except when comparing results to those of general relativity. λ and h are dimensionless coupling constants. The factor $1/12$ in the $R^\mu{}_\mu$ term is necessary for the resulting field equations to be conformally invariant (see, for example, Birrell and Davies [4]). We shall not be concerned with fermion fields in this paper. Omitting them, this action yields the field equation for S (PM, equation (63)):

$$S^{;\mu}{}_\mu + \frac{1}{6} S R^\mu{}_\mu - 4\lambda S^3 = 0 \quad (4)$$

and the expression for $T_{\mu\nu}$ (PM, equation (64)):

$$T_{\mu\nu} = \frac{2}{3} S_{;\mu} S_{;\nu} - \frac{1}{6} g_{\mu\nu} S^{;\alpha}{}_{;\alpha} - \frac{1}{3} S S_{;\mu;\nu} + \frac{1}{3} g_{\mu\nu} S S^{;\alpha}{}_{;\alpha} - \frac{1}{6} S^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha{}_\alpha \right) - g_{\mu\nu} \lambda S^4 \quad (5)$$

II. APPLICATION TO COSMOLOGY

We assume that on the largest scales our Universe can be adequately described as a Friedmann-Robertson-Walker (FRW) space. Adapting the field equations to such a space results in the Mannheim model, the main features of which are recapitulated in this section.

The metric tensor for FRW space can be written

$$g_{\mu\nu} = \text{Diag} \left[-1, \frac{R^2(t)}{1 - kr^2}, R^2(t)r^2, R^2(t)r^2 \sin^2 \theta \right] \quad (6)$$

where $R(t)$ is the expansion factor and k can take the values 1, 0, or -1 . Mannheim opts for $k = -1$ in his model. From this metric tensor we can derive an expression for $R^\alpha{}_\alpha$:

$$R^\alpha{}_\alpha = -\frac{6}{R^2} (k + \dot{R}^2 + R\ddot{R}) \quad (7)$$

Since a FRW space is conformally flat, the Weyl tensor vanishes, and the field equations (1) reduce to (PM, equation (219)):

$$T^{\mu\nu} = 0 \quad (8)$$

Mannheim assumes a conformal transformation is made that reduces the field S to a constant value, S_0 . The field equation (4) then requires

$$S_0^2 = \frac{1}{24\lambda} R^\alpha{}_\alpha \quad (9)$$

and (8) becomes, using (5),

$$T_{\text{kin}}^{\mu\nu} - \frac{1}{6} S_0^2 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha{}_\alpha \right) - \lambda S_0^4 g^{\mu\nu} = 0 \quad (10)$$

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where $T_{\text{kin}}^{\mu\nu}$ is the energy-momentum tensor of any matter fields. (In this paper, since we neglect fermion fields, we assume this tensor derives entirely from electromagnetic radiation.)

Writing (10) in the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_{;\alpha}^{\alpha} = \frac{6}{S_0^2}(T_{\text{kin}}^{\mu\nu} - \lambda S_0^4 g^{\mu\nu}) \quad (11)$$

we see that the field equations are just those of general relativity, but with an effective gravitational constant that is *negative* (PM, equation (224)):

$$G_{\text{eff}} = -\frac{3c^3}{4\pi S_0^2} \quad (12)$$

and a cosmological constant

$$\Lambda = \lambda S_0^4 \quad (13)$$

Mannheim chooses $\lambda < 0$, and assumes $T_{\text{kin}}^{\mu\nu}$ consists in radiation with a density $\rho_m = A/R^4(t)$. He then solves the Friedmann equation derived from the field equations (11) to get (PM, equation (230)):

$$R^2(t) = -\frac{k(\beta-1)}{2\alpha} - \frac{k\beta \sinh^2(\alpha^{1/2}ct)}{\alpha} \quad (14)$$

where

$$\alpha c^2 = -2\lambda S_0^2 = \frac{8\pi G_{\text{eff}}\Lambda}{3c} \quad (15)$$

$$\beta = \left(1 - \frac{16A\lambda}{k^2c}\right)^{1/2} \quad (16)$$

Note that, since $\lambda < 0$, α is positive and $\beta > 1$.

According to (14), $R(t)$ does not go to zero as $t \rightarrow 0$, but to the finite value $[(\beta-1)/(2\alpha)]^{1/2}$.

Mannheim argues (PM, page 426) that this solution may hold the key to the cosmological constant problem, since the quantity

$$\bar{\Omega}_\Lambda(t) \equiv \frac{8\pi G_{\text{eff}}\Lambda}{3cH^2(t)} \quad (17)$$

goes like $\tanh^2(\alpha^{1/2}ct)$ for large t , and hence is bounded from below no matter how large Λ may be.

III. BREAKING THE CONFORMAL SYMMETRY

The Mannheim model has some attractive features, but a basic drawback in that the effective gravitational constant, G_{eff} , is negative. One can argue that this negative value is only apparent on the largest scales, but it seems quite unlikely that, for example, the observations of the cosmic microwave background (CMB) can be reconciled with such a G_{eff} .

In this section we introduce ideas that are familiar from inflation and particle physics, and try to modify Mannheim's procedure so as to achieve a G_{eff} that is positive. The simplest solution to Mannheim's field equations is Minkowski space, where $k = 0$ and $\lambda = 0$. Since the space is flat, R^α_α is also zero, and (4) can be satisfied with an arbitrary value of S_0 .

Let us suppose that a space of this sort is metastable against the formation of bubbles of lower symmetry. Once a bubble forms it will expand at the velocity of light, and the geometry inside the bubble is of FRW type, with $k = -1$ (the idea of such bubble formation goes back to 1980 [5], and has been discussed many times since, for example in [6]).

We will assume the field equations are still given by (1), but the action (3) is replaced by

$$I_M = - \int d^4x (-g)^{1/2} \left\{ \frac{1}{2} S^{;\mu} S_{;\mu} - \frac{\mu}{12} S^2 R^\mu_\mu + \lambda S^4 + i\bar{\psi}\gamma^\mu(x) [\partial_\mu + \Gamma_\mu(x)] \psi - h S \bar{\psi}\psi \right\} \quad (18)$$

where the coefficient μ in the R^μ_μ term need not have the value unity, so general conformal invariance is broken. We still, of course, have invariance under the restricted conformal transformation

$$g_{\mu\nu} \rightarrow e^{2\alpha} g_{\mu\nu} \quad (19)$$

with α a constant, and no longer a general function of x^μ .

We will not concern ourselves with details of the bubble formation, and the reader may prefer simply to start from the field equations (1) and the action (18).

Omitting fermion fields, the field equation for S becomes (see (4)):

$$S^{;\mu}_\mu + \frac{\mu}{6} S R^\mu_\mu - 4\lambda S^3 = 0 \quad (20)$$

and the expression for $T_{\mu\nu}$ (see (5)):

$$T_{\mu\nu} = \frac{2}{3} S_{;\mu} S_{;\nu} - \frac{1}{6} g_{\mu\nu} S^{;\alpha}_\alpha - \frac{1}{3} S S_{;\mu;\nu} + \frac{1}{3} g_{\mu\nu} S S^{;\alpha}_\alpha - \frac{\mu}{6} S^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\alpha_\alpha \right) - g_{\mu\nu} \lambda S^4 \quad (21)$$

In a FRW space we can assume S is a function of t alone, and the equation of motion becomes

$$\ddot{S}R^2 + 3\dot{S}R\dot{R} + \mu k S + \mu S\dot{R}^2 + \mu S R\ddot{R} + 4\lambda S^3 R^2 = 0 \quad (22)$$

Two independent equations result from $T_{\mu\nu} = 0$; we take them to be the (0 0) and (1 1) components:

$$R^2 \dot{S}^2 + 2S\dot{S}R\dot{R} + \mu k S^2 + \mu S^2 \dot{R}^2 + 2\lambda S^4 R^2 = 0 \quad (23)$$

$$4S\dot{S}R\dot{R} - R^2 \dot{S}^2 + 2R^2 S\ddot{S} + \mu k S^2 + \mu S^2 \dot{R}^2 + 2\mu S^2 R\ddot{R} + 6\lambda S^4 R^2 = 0 \quad (24)$$

Now (23) can be derived from (22) and (24), so we actually only have two independent equations for R and S , which we can take to be (22) and (24). But this turns out to be a singular pair. If we were to try to integrate them numerically, we would have to rearrange them into the form

$$\ddot{R} = f_1(R, S, \dot{R}, \dot{S}) \quad (25)$$

$$\ddot{S} = f_2(R, S, \dot{R}, \dot{S}) \quad (26)$$

but we find easily that such a rearrangement is impossible.

We can, however, solve these equations by simply choosing for S the solution $S = S_0 = \text{constant}$. Having done that, we can solve for R , then check back to ensure that our solution is consistent with our choice for S . We do not need to invoke a conformal transformation to do this; we just have to specify S before we can solve for R . (Is $S = \text{constant}$ the only consistent choice? We do not know.)

The calculation now follows the same pattern as Mannheim's. We get a Friedmann equation of the same form, except for the inclusion of a factor μ , so we have to solve

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_{;\alpha}^{\alpha} = \frac{6}{\mu S_0^2} (T_{\text{kin}}^{\mu\nu} - \lambda S_0^4 g^{\mu\nu}) \quad (27)$$

which gives

$$G_{\text{eff}} = -\frac{3c^3}{4\pi\mu S_0^2} \quad (28)$$

This will be positive if we choose $\mu < 0$. The reversal of sign of the S^2 term in the action (18) has been noted before in connection with ϕ^4 field theory, which the Mannheim model resembles; see, for example, [7] – [9].

We will also reverse the sign of λ , so it is now positive. A few words need to be said about this choice also. As noted, the field equation (4) for S resembles that of the scalar field, ϕ , in conventional ϕ^4 theory. In particular, $\lambda > 0$ is the “right-sign” choice, as illustrated in [7] – [9]. This insures a spectrum that is bounded below. Choosing $\lambda < 0$, as Mannheim does, entails treating the theory as a PT symmetric one [10].

Introducing radiation with energy density $\rho = A/R^4(t)$, the Friedmann equation becomes (PM, equations (225) and (226)):

$$\dot{R}^2 = \frac{8\pi G_{\text{eff}} A}{3c^2 R^2} - kc^2 + \frac{8\pi G_{\text{eff}} \Lambda R^2}{3c} \quad (29)$$

Here $\Lambda = \lambda S_0^4$ and $k = -1$, as before.

Provided A is not too large, this has an exact solution that can be written in a notation analogous to that of

Mannheim:

$$R^2(t) = \frac{(\beta - 1)}{2\alpha} + \frac{\beta}{\alpha} \sinh^2 \left[\alpha^{1/2} c (t + \gamma) \right] \quad (30)$$

$$\alpha = -\frac{2\lambda S_0^2}{\mu} > 0 \text{ for } \mu < 0 \text{ and } \lambda > 0 \quad (31)$$

$$\beta = \left[1 - \frac{16A\lambda}{\mu^2 c} \right]^{1/2} < 1 \text{ for } \lambda > 0 \quad (32)$$

$$\gamma = \frac{1}{\alpha^{1/2} c} \operatorname{arcsinh} \left[\left(\frac{1 - \beta}{2\beta} \right)^{1/2} \right] \quad (33)$$

With this choice for γ , $R(t) \rightarrow 0$ as $t \rightarrow 0$. From (32) we obtain the upper limit for A for this analytic solution to be valid:

$$A \leq \frac{\mu^2 c}{16} \quad (34)$$

Using (7) we can show

$$R^\alpha_{;\alpha} = \frac{24\lambda S_0^2}{\mu} \quad (35)$$

so that that (20) is satisfied when $S = S_0$, and we have a consistent solution to our field equations.

IV. COMPARISON WITH THE MANNHEIM MODEL

Points of agreement: the two models coincide for large t , when the Λ term dominates. Further, if Mannheim is correct in claiming that his model solves the cosmological constant problem, then the same can be said for the present model, since $\bar{\Omega}_\Lambda$ has a similar behavior. Both models have $k = -1$.

Points of difference: the main one, of course, is that the present model has a positive G_{eff} . As a consequence, $R(t)$ tends to zero, rather than a non-zero value, as $t \rightarrow 0$.

V. CONCLUSION

By breaking the conformal symmetry in Mannheim's model, we have constructed a new model with a positive G_{eff} . This model has an attractive feature of the Mannheim model (possible solution of the cosmological constant problem), and at the same time is sufficiently similar to conventional general relativity that it may prove capable of explaining cosmological observations.

If we assume the field equations (1) remain the same, with the new $T^{\mu\nu}$, then one of the main achievements of conformal gravity, the interpretation of galactic rotation curves, is unaltered. Cosmology has nothing to say about the coupling constant α_g , since the Weyl tensor vanishes in a FRW space. We note, however, that if we are to obtain the pleasing behavior of the gravitational potential described in [11], α_g must be taken to be positive.

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